Experimental Analysis of the Superposition of Information in Random Memory Spaces

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Abstract. In this work a model of information storage is presented, it is specially adapted for not supervised neurocomputing systems. Our model uses the Ramsey Theorem as theoretical foundation. Experimentally is shown that a discrete matrix randomly generated and sufficiently large (dependent of $m \times n$), is possible to find any discrete matrix of size $m \times n$. The probability $\rho(\varphi \subset M)$ of finding a sub-matrix of size m x n within a specific Random Memory Space increases when limited degree of error is allowed. Latter w Then we introduce the concept of information pattern. We also applied different linear transformations to the original matrix, which extends the search space and therefore also increases the probability $\rho(\varphi \subset M)$. The model is implemented using memories of four states and is demonstrated one of the main characteristics: the information superposition. A physical element of memory is used to store several patterns of information at the same time. It is shown that for information patterns of square dimension ($m \times m$) the maximum degree of superposition that can be obtained is $(2m - 1)^2$ and that in an RMS of relatively small size is possible to store a large amount of information patterns.

1 Introduction

The necessity of models to represent large amounts of information is characteristic in the neurocomputing field [2]. The goal of this work is to raise a model of information storage: Random Memory Space (RMS). This model has theoretical foundation in the Ramsey theorem, which mainly affirms that if a graph contains sufficient number of vertices (lets say, dependent of k), then it must contain a complete set or an independent set of size k [4].

Our model starts off from the supposition of within a discrete matrix of size $i \times j$ sufficiently large and with randomly values generated, it is possible to find any discrete matrix of size $m \times n$, with a probability directly proportional to the size of the random matrix and inversely proportional to the size of the submatrix that looks for.

In general, the model consists in a binary matrix M randomly generated. The information that is tried to search and store is put under a transformation, by means of which a binary chain s becomes a matrix φ. Takes place a search of φ in each one of the defined linear transformations for matrix M, until is found a submatrix Θ of M that is equal to φ. In that moment a memory element is "marked" which has a specific position within Θ with a reference to the transformation in which the submatrix was found. This allows the effective recovery of all the matrices φ and therefore of all the "stored information".

Since the use of the elements of a submatrix Θ by a matrix φ_1 does not prevent that some of these elements are used by some other matrices $\varphi_2, \varphi_3, \ldots, \varphi_n$, is possible that several matrices φ_i share the same physical memory space, giving rise to the information superposition (Figure 1).

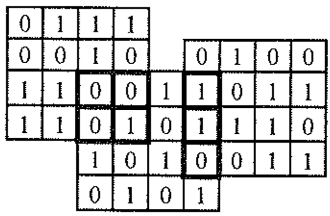

Fig. 1. Information superposition in a random memory

If we allowed that the search of matrices φ in the Random Memory Space present limited errors, this is, if we allowed that a limited difference exists between φ and some submatrix Θ, then the probability of finding Θ within M increases.

We are speaking then of information patterns, since our interest is not centered in an exact representation of an information structure but in a fundamental structure that must be conserved and that it is significant within some computing paradigm. In section 5 we experimentally show that introducing the concept of information pattern dramatically increases the probability of finding within *M* some Θ that it conserves the fundamental structure of a matrix φ.

Although the calculation of the size of the random matrix for a specific size of submatrices *m x n* remains unknown, we set out:

1. To find experimentally a relation between the sizes of the matrices, in such a way that we have the certainty that given a size of submatrix, will be possible to find any instance of this size within the random matrix;
2. Studying the behavior of the model when it is tried to store large amounts of information and
3. Proposing the model as an effective and efficient paradigm for neurocomputing systems.

2 Ramsey Theorem

In the mathematics field there are several theorems that affirm in a general form, that all system of a certain class contains a subsystem with a degree of organization greater than the original/container system [1].

The known "Ramsey Theorem" type theorems prove the above affirmation using different mathematic objects of analyses: sets and graphs (Ramsey) [7], equations (Schur, Rado) [8] [9], arithmetical progressions (Van der Waerden) [12], finite sequences constructed from finite sets (5 Hales-Jewett) [5], vectorial spaces (Graham-Lib-Rothschild) [3], etc. These theorems conform what it is known in a general way like the Ramsey Theory, a discipline within the discrete mathematics. Most of these theorems affirm that a colored in r of any structure sufficiently large contains a monochrome substructure of certain size. In terms of the graph theory, if a graph contains sufficient vertices (a dependent number of *k*), then it must contain either a complete set or an independent set of vertices of size *k*.

It turns out advisable to mention that some mathematical theorems as Bolzano-Weierstrass, affirms that within any limited sequence of complex numbers, a convergent subsequence exists, fall within the class of theorems mentioned in [1], but do not conform part of the Ramsey Theory.

In order to give a formal definition we introduce the following notation.

$$\text{Sea } Z^+ = \{1,2,\ldots\} = \text{the positive integers}$$

$$I_n = \{1,\ldots,n\},\ n \in Z^+ \text{ an arbitrary set of cardinality n} \tag{1}$$

$$[A]^k = \{B : B \subset A, |B| = k\}$$

A coloring r on a set S is mapped

$$f: S \to I_r \tag{2}$$

For $s \in S$, f(s) asigned the color of s ; we say that f is monochrome low, if $f(s)$ is constant in T, for a set $T \subseteq S$, this is:

$$f(t) = r_i \quad \forall\, t \in T,\ T \subseteq S, \text{ i constant} \tag{3}$$

Given any colored r of $[n]^2$, if $\exists\, i$, $1 \le i \le r$, and a set $T \subseteq [n]$, $|T| = l_i$ so that $[T]^2$ is monochrome in i, then we wrote

$$n \to (l_1, \ldots, l_r) \tag{4}$$

The function of Ramsey $R(l_1,\ldots,l_r)$ denotes the minimum value of n so that the previous proposition is true.

The generalization of the previous case is when we considered colored r of $[n]^k$, where k is an arbitrary integer number.

We define $n \to (l_1,\ldots,l_r)^k$ if, for all colored r of $[n]^k$, $\exists\, i$, $1 \le i \le r$, and a set $T \subseteq I_n$, $|T| = l_i$ so that $[T]^k$ is monochrome in i.

In this case, the function of Ramsey for sets of k-cardinality is indicated by R_k.

$$R_k(l_1,\ldots,l_r) = \min\{n_0 : \text{for } n \ge n_0,\ n \to (l_1,\ldots,l_r)^k\} \tag{5}$$

The Ramsey theorem affirms that the function R_k is well defined; this is, $\forall$ k, $l_1,\ldots,l_r$, exists n_0 so that for $n \ge n_0$ is fulfilled that

$$n \rightarrow (l_1,\ldots,l_r)^k \qquad (6)$$

Several demonstrations of this theorem have been developed. The original demonstration in charge of Frank P. Ramsey takes the case for an infinite set like and the finite set [7].

The calculation of the exact values for the Ramsey function R(k,l) for small values of k, l has took tremendous efforts, nevertheless, until now only the exact values of the superior and inferior levels are known, published in [6].

Our interest is to prove experimentally the basic idea of the Ramsey Theorem in discrete matrices and using this powerful idea as foundation for a new paradigm of information storage in neurocomputing systems. Because of it, we must prove experimentally that in a discrete matrix M sufficiently large (possibly generated in random way) we can find any discrete matrix φ of a smaller size than M, with a probability that increases as the size of M also increases, in other words $\rho(\varphi \subset M)$ increases if M increases its size

3 The Formal Model

We defined a Random Memory Space (RMS) as the tuple $<M,E,S,T,\Theta,\Gamma,\varepsilon>$ where M is a bidimensional matrix, $E \subset Z^+$, S is an ordered pair, T is a set of τ linear transformations on M, M_v y M_r, Θ is the set submatrices of size *m x n* contained within $\tau(M_v)$ and Γ it is the set of patterns that are store in the RMS are sets, and $\varepsilon \in Z^+$

M is a matrix of size *i x j* that contains structures of the form (v,r) where $v \in E$, r is a reference to some $\tau \in T$, E and T are finite sets and

$$E \subset Z^+ \qquad (7)$$

We define the set of values E' as

$$E' = \{e : e \subset E \ \wedge |E'| = |\hat{E}| / 2 \ \} \qquad (8)$$

The v values are initially generated in a random way and obey the constrain

$$v \in E' \qquad (9)$$

We define the matrix of values M_v just like the one we gets from deleting the r element r of each structure (v,r) of the matrix M. The references matrix M_r is the one we gets from deleting the v element of each structures (v,r) of the matrix M.

The relation between M_{vij} element and M_{rij} element is permanent, this means, and M_{vij} element is related only with the M_{rij} element in any given moment.

$$S = (m,n) \text{ where } m,n \in Z^+ \qquad (10)$$

Let M^τ, M^τ_v y M^τ_r generating the matrix sets for the set of τ linear transformations over M, M_v y M_r,

$$M^{\tau} = \{\tau(M) : \tau \in T\} \qquad (11)$$

$$M^{\tau}_{v} = \{\tau(M_v) : \tau \in T\}$$

$$M^{\tau}_{r} = \{\tau(M_r) : \tau \in T\}$$

Θ is the set submatrices of size *m x n* contained within $\tau(M_v)$ matrices of M^{τ}_{v}

$$\Theta = \{\Theta_{xy} : \Theta_{xy} \subset \tau(M_v) \text{ for some } \tau\} \qquad (12)$$

Where each Θ_{xy} element within Θ is formed by the set of P_{xy} elements

$$P_{xy} = \{P_{xy11}, P_{xy12}, \ldots, P_{xyhk}, \ldots, P_{xymn}\} \qquad (13)$$

And where each P_{xyhk} is a v value in $\tau(M_v)$ with a position (h,k) given a relative origin (x,y).

Given some $\Theta_{x+a\ x+b} \in \Theta$, where a, b $\in Z^{+}$, if the conditions $|a| < m$ y $|b| < n$ fulfill, and

$$B = \Theta_{xy} \cap \Theta_{x+a\ x+b} \qquad (14)$$

then the next proposition is true:

$$B \neq 0 \qquad (15)$$

It can be observed that the submatrices Θ_{xy} and $\Theta_{x+a\ x+b}$ contain P_{xyhk} elements in common within some matrix $\tau(M_v)$. The amount of elements shared within $\tau(M_v)$ by this pair is

$$|B| = [\,abs(m - abs(a))\,]\ [\,abs(n - abs(b))\,] \qquad (16)$$

where abs(x) is the absolute value of x.

The number of submatrices Θ_{xy} in M^{τ}_{v} is denoted by

$$c = [(i - m + 1)(j - n + 1)]\, t \qquad (17)$$

where t is the number of linear transformations applicable to M.

$$T = |T| \qquad (18)$$

An input pattern φ is a binary matrix of size *m x n*. Applying the τ linear transformations to the M_v matrix increases the occurrence probability of φ

$$\rho(\varphi \in \Theta) \qquad (19)$$

The search of a pattern φ in the RMS is mapping from the input plane φ to some plane $\tau(M_v)$ for some τ, resulting a submatrix $\Theta_{xy} \in \Theta$, which no necessary is equal to φ, and the number of elements which they differ is denoted by

$$\varepsilon(\varphi, \Theta_{xy}) = \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \varphi_{(k,l)}\ \psi\ \Theta_{xy\ (x+k,\ y+l)} \qquad (20)$$

where ψ is an equivalent operator, that is defined as follows:

$$a\,\varphi\,b = \begin{cases} 0 & \text{if aRb} \\ 1 & \text{otherwise} \end{cases} \quad a \in E',\ b \in E \tag{21}$$

The relation R: E' → E is reflexive and keeps correspondency 1 to 2.

For the input pattern φ can be assigned to the submatrix Θ_{xy}, the following conditions are needed

$$\varepsilon\,(\varphi,\Theta_{xy})\ P\ \varepsilon \quad y \quad \Theta_{xy11} \in E' \tag{22}$$

due to the moment of assignment, the v value of Θ_{xy11} is replaced with $w \in E$, vRw, $v \neq w$.

and this element will not be able to be used to mark any other input pattern φ_1. Aditionally, to the r element, with which Θ_{xy11} forms a structure (v,r) in M, must be assigned to it a referente to the transformation τ for wich Θ_{xy} is a submatrix of $\tau(M_v)$.

Let Γ it is the set of patterns that are store in the RMS given a set ϑ of input patterns

$$\Gamma = \{\Theta_{xy} : (\sum_{k=0}^{m-1}\sum_{l=0}^{n-1} \varphi_{(k,l)}\ \psi\ \Theta_{xy\,(x+k,\,y+l)})\ T,\ \varepsilon, \tag{23}$$

$$\Theta_{xy11} \in (E - E'),\ \Theta_{xy} \subset \tau(M_v) \text{ para algún } \tau \in T,\ \varphi \in \vartheta\}$$

The degree of superposition of a M_{ij} element, this is, a structure (v,r) with position *i, j* in matrix M, is the number of submatrices Θ_{xy} that store some input pattern φ with $\varepsilon(\varphi,\Theta_{xy}) <= \varepsilon$ and that shares the M_{vij} element of M_v. For input patterns φ with square geometry *m x m* the maximum degree of superposition is determined by

$$(2m-1)^2 \tag{24}$$

Because when storing a pattern, an element (v,r) is "marked", the maximum number of patterns that can be stored in a random memory space is *i x j*, the size of matrix M.

4 Implementation

M is implemented like a vector of n matrices $(M_1,M_2,\ldots,M_n)$ of size *i x j*. Since we wished to handle binary patterns of information, each element of M_i is a memory unit of 4 states:

$$E = (0, 1, 0^*, 1^*)\ \ y\ \ E'\ (0, 1) \tag{25}$$

The value of n depends on the number of transformations τ defined on M, M_v and M_r ($t = |T|$). In our experiments we decided to use two sets of linear transformations. The first one consists of 16 transformations that are obtained as the following way:

1. 4 different rotations: 90, 180 and 270 degrees.
2. To each one of these rotations, transferring in its horizontal axis.

3. To each one of the 8 previous transformations, invert the logical values of the matrix.

Since each element of memory can codify four different states, a vector of two elements can codify sixteen transformations. We say that

$$M_v = M_1$$
$$M_r = (M_2, M_3) \tag{26}$$

and an element (structure) of M with position i,j is defined by

$$M_{ij} = (M_{1ij}, (M_{2ij}, M_{3ij})) \tag{27}$$

where $v = M_{1ij}$ y $r = (M_{2ij}, M_{3ij})$.

The second set includes 48 transformations, 3 of which we called basic:

1. The original matrix.
2. Interchanging even of lines.
3. Interchanging even of columns.
4. To each one of the three previous transformations, 16 transformations of the first set are applied to it.

In this case, we defined four matrices: M_1, M_2, M_3 and M_4:

$$M_v = M_1$$
$$M_r = (M_2, M_3, M_4) \tag{28}$$
$$M_{ij} = (M_{1ij}, (M_{2ij}, M_{3ij}, M_4))$$

The size $i \times j$ of M, the size $m \times n$ of the input patterns φ and the allowed error degree ε in the search of patterns within M_v, stay as parameters for the experimental analysis.

However, in our experiments we restricted the dimensions of M to m square spaces, this is, $i = j$ and $m = n$.

5 Experimental Analysis

The first experiment consisted of measuring the occurrence frequency of the elements inside a set of generated randomly input patterns.

The objective is to find experimentally a relation between the sizes of a matrix M, the size of the patterns φ and the allowed degree of error, in such a way that we have the certainty that any input pattern will be found inside a specific Random Memory Space.

The results are in Tables 1 and 2 respectively with 16 and 48 transformations of RMS's.

Next we measured one of the most important characteristics of the model: the information superposition. For it we generate randomly n patterns, carried out the search and allocation of each pattern and later we measured, for each structure (v,t) in M, how many patterns are occupying this structure (the superposition degree). The value of n must be adequate in order to find all the patterns within the RMS. Tables 3 and 4 show the number of structures that present certain degree of superposition given the size of a RMS, a size of input pattern, a number n of patterns, and an specific error degree ε. These results are the obtained as average of 50 repetitions of the experiment for 16 transformations of RMS's.

Table 5 shows the results obtained for a RMS of 120x120. It is used exclusive patterns of size 3x3 and 4x4 because it is not possible to find all the patterns generated when its size is 5x5.

Table 1. Experimental results of the occurrence frequency of a binary pattern within a RMS using 16 transformations (averages on 50 patterns)

Pattern Size	Error Degree	RMS Dimension			
		120x120	300x300	480x480	960x960
3x3	0	437	2772	7126	28668
	1	4327	27766	71356	286715
	2	20023	127789	328388	1319594
4x4	0	3	21	54	223
	1	55	365	944	3799
	2	460	2961	7640	30607
	3	2336	14991	38754	155846
5x5	0	0	0	0	0
	1	0	1	3	9
	2	2	12	35	145
	3	16	111	282	1166
	4	96	639	1651	6639

Table 2. Experimental results of the occurrence frequency of a binary pattern within a RMS using 48 transformations (averages on 50 patterns)

Pattern Size	Error Degree	RMS Dimension			
		120x120	300x300	480x480	960x960
3x3	0	1307	8341	21445	86008
	1	12983	83264	214137	859308
	2	59928	383249	984917	3957367
4x4	0	10	65	166	673
	1	171	1097	2824	11338
	2	1394	8825	22867	91888
	3	7036	45029	116272	466911
5x5	0	0	0	0	2
	1	0	3	9	33
	2	5	38	107	430
	3	50	325	855	3461
	4	297	1923	4973	19963

Table 3. Results of information superposition (IS) in random matrices of size 120x120 with 16 transformations

Pattern Size	Error Degree	Number of Patterns	Superposition Degrees										
			0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	10°
3x3	0	250	13108	707	321	181	61	18	4	0	0	0	0
		500	12503	740	437	298	213	133	58	17	1	0	0
		1000	11426	939	553	450	297	304	226	115	74	16	0
4x4	1	250	11306	2366	572	136	18	2	0	0	0	0	0
		500	9074	3445	1305	414	118	35	7	2	0	0	0
		1000	6571	3676	1941	1144	591	293	122	48	13	1	0
5x5	2	250	10101	3335	845	91	8	0	0	0	0	0	0
		500	6690	5070	2006	545	87	2	0	0	0	0	0
		1000	3417	4705	3456	1822	733	210	42	12	3	0	0

Table 4. Results of IS in random matrices of size 300x300 with 16 transformations

Pattern Size	Error Degree	Number of Patterns	Superposition Degrees									
			0°	1°	2°	3°	4°	5°	6°	7°	8°	9°
3x3	0	2000	9833	1120	667	607	408	376	417	334	343	290
		8000	1893	1606	1201	1119	792	745	931	836	1170	4099
		12000	60	175	315	683	663	804	1121	1154	1645	7343
4x4	1	2000	3717	3242	2261	1598	1210	945	656	409	216	92
		8000	112	271	462	699	922	950	1030	1089	1093	1045
		12000	5	26	63	117	241	116	122	190	353	407

Pattern Size	Error Degree	Number of Patterns	Superposition Degrees									
			10°	11°	12°	13°	14°	15°	16°	17°	18°	19°
3x3	0	2000	0	0	0	0	0	0	0	0	0	0
		8000	3	0	0	0	0	0	0	0	0	0
		12000	371	50	9	1	0	0	0	0	0	0
4x4	1	2000	35	9	2	0	0	0	0	0	0	0
		8000	1062	1134	1190	1196	1075	729	294	35	1	0
		12000	611	896	1239	1560	2060	2590	2507	943	249	62

Pattern Size	Error Degree	Number of Patterns	Superposition Degrees					
			20°	21°	22°	23°	24°	25°
3x3	0	2000	0	0	0	0	0	0
		8000	0	0	0	0	0	0
		12000	0	0	0	0	0	0
4x4	1	2000	0	0	0	0	0	0
		8000	0	0	0	0	0	0
		12000	18	10	8	3	2	1

Table 5. Results of IS in random matrices of size 120x120 with 16 transformations and a number of patterns near to storage the Maxima capacity

Pattern Size	Error Degree	Number of Patterns	Superposition Degrees										
			0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	10°
3x3	0	500	87668	1075	660	363	171	48	13	2	0	0	0
		1000	86893	986	578	533	341	321	203	96	41	8	0
		2000	85335	992	643	775	498	460	550	321	276	150	0
4x4	1	500	84458	3805	1226	354	111	39	7	0	0	0	0
		1000	81650	4207	2090	1099	586	261	83	23	1	0	0
		2000	78251	4366	2371	1658	1259	850	558	376	185	97	21
5x5	2	500	78338	10851	784	27	0	0	0	0	0	0	0
		1000	68467	18443	2744	3165	31	0	0	0	0	0	0
		2000	52005	28124	8016	1610	215	26	4	0	0	0	0

6 Result Analysis

In Tables 2 and 3 we can observe how the probability of finding a input pattern within a RMS of particular size increases directly proportional to the size of the RMS and the allowed degree of error and inversely proportional to the size of the input pattern.

Tables 3, 4 and 5 along with Figures 2 and 3 reveal in very clear form the information superposition that takes place in each element of physical memory within a RMS.

We can observe quantitative and qualitatively how a same physical element of storage can be used to store more information of what is capable a conventional model of memory. In Section 3 is affirm that the maximum degree of possible superposition when we used input patterns with square geometry $m \times m$ is $(2m - 1)^2$, this means, it is possible that $(2m - 1)^2$ patterns can use the same element of physical memory.

Actually we see that the maximum degree of superposition is much smaller, even though the number of stored patterns is very near to the capacity of the RMS, this is because the calculation of the maximum degree of superposition obeys to a specific distribution of the patterns in the different transformations from the matrices and the position within these transformations, distribution that does not have any certainty to be obtained given the random nature of the matrix.

We can perform an analysis of the theoretical capacity of information storage in a particular RMS and compare this value with its analogous, using a conventional model of memory. This is, in a RMS of $i \times i$ that stores binary input patterns of size m x m (using physical memories of four states), theoretically we can store $i \times i$ patterns, therefore we can store $i^2 \times m^2$ bits, this is, it is possible to be codified $2^{i \times i \times m \times m}$ combinations. When the same space of physical memory is used in a conventional way we can store to $i \times i$ elements of memory of 4 states, this is, $2 \times i^2$ bits, with which they are possible to be codified $2^{2 \times i \times 1}$ combinations. The difference is clear, and can be observed that the capacity of storage remarkably is increased when increasing the size of the input pattern. For example, if the size of the input patterns increases to $(m+1) \times (m+1)$ the amount of combinations that can be obtained is $2^{i \times i \times (m+1) \times}$

$(m+1)$ the amount of combinations that can be obtained is $2^{i \times i \times (m+1) \times (m+1)}$ which means $2^{i \times i \times (2m+1)}$ times more than the quantity obtained with patterns of size $m \times m$.

However, it is not possible to increase capriciously the size of the input patterns; it must obtain a balance between the capacity of storage and the probability of finding these patterns.

7 Conclusions

The model of Random Memory Spaces for the information storage presents several interesting characteristics, between which they stand out its primarily random nature and the information superposition. At the present time the paradigm of the quantum computation uses the concept of superposition of states using abstract entities called qbits as fundamental element of storage [13]. Apparently the use of a same physical organization to represent different things (states, information, etc.) of simultaneous way, it is an important concept for the development of new computing paradigms.

The direction towards the processes of storage and the simple manipulation of information is a characteristic that distinguish to the Random Memory Spaces of other models of storage trims in the direct representation of information. Nevertheless, these simple processes and manipulations provide a great capacity of dynamic storage of information.

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